Vector fields and flows

- representing fns on a submfld
- tangent spaces

Finishing lec 3

Quotients

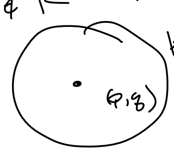
Set theory: If X is a set, $R \subseteq X \times X$ is an equivalence relation if
$(x,x) \in R \ \forall x$
$(x,y) \in R,\ (y,z) \in R \Rightarrow (x,z) \in R$
$(x,y) \in R \Rightarrow (y,x) \in R$.
kinda write $x \sim y$ for $(x,y) \in R$.

The quotient X/R is X, but with $x = y$ if $(x,y) \in R$.
$\pi : X \to X/R$

Quotient topology

$U \subseteq X/R$ is open iff $\pi^{-1} U$ is open in X.

Prop if X is a manifold (more generally: Haus + loc cpct)
then X/R is Hausdorff iff X is closed.
idea $(p,q) \notin R \Rightarrow \exists\ U \times U \subseteq X \times X$ separating p from R

T  K . $R \cap K$ cpct

S $\quad U \overset{\text{open}}{\subseteq} \mathbb{R}^n$

Quotient mfld

Let S be a manifold (just, we know what $f : U \to S$, $f : S \to U$ smooth means)
has a notion of smooth maps
• For any R, S/R
in and out (cf. subsets):

Dfn. • $f : S/R \to U$ smooth if $S \to S/R \to U$ smooth
• $f : U \to S/R$ smooth if it locally lifts to S

$$\begin{array}{ccc} & & S \\ & \nearrow & \downarrow \\ U' & \to & S/R \end{array}$$

is an equiv rel'n on S,
dim r

Thm If R, is a submfld of $S \times S$ and $\pi_1 : R \to S$ is a submersion,

Then the top space S/R has a smooth atlas of dim $d-r$

Corrected proof:

Suppose $p \in S$. Let $N \overset{\dim r}{\subseteq} S$ be s.t. $p \times N$ is a transversal to $R \cap p \times S$ (using π_1 a submersion).

Again by π_1 submersion, $R \cap N \times S$ is a manifold of dimension d, containing (p,p). Furthermore, one checks that $d\pi_2 : T_{(p,p)} R \cap N \times S \longrightarrow T_p S$ is an isomorphism, so π_2 is locally invertible. Then $\pi_1 \circ \pi_2^{-1}$ and the inclusion ι_N give inverse smooth maps, representing a chart for S at p

Conclusion

If S mfld, R submfld, π_1 subm, and R closed, then

S/R is an abstract manifold.

Thm (Whitney, part 2) Every compact (Hausdorff) abstract d-mfld embeds into $\mathbb{R}^n$ for some n (applies also to open subsets w/ cpt closure)

RE PoU + coord fns & compactness

<u>Together</u> Every abstract mfld embeds in $\mathbb{R}^n$

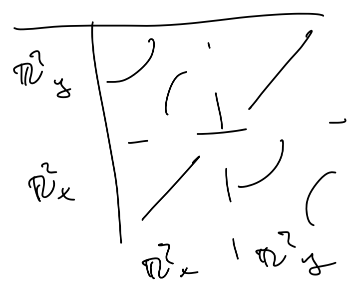
<u>Conclusion</u> 3 ways to prove something is a mfld.

- prove it's an abstract mfld
- prove it's realizable as a submfld of $\mathbb{R}^n$

(*) • prove it is constructed from other manifolds by
  - Finite products
  - Embeddings (inc. open subsets)
  - Level sets
  - Countable disjoint unions
  - Quotients by R as in conclusion.

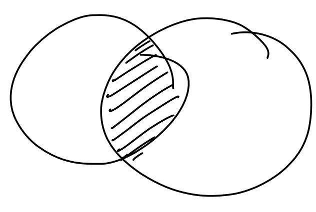
<u>Examples of quotients</u>

- $S^1 = \mathbb{R}_x \sqcup \mathbb{R}_y / R$ where R is generated by $x \sim \frac{1}{y}$

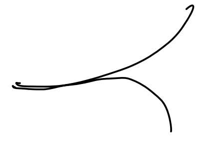


<u>closed?</u> yes. graph of $\frac{1}{y}$ closed

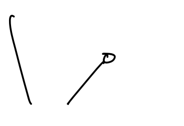
Picture



$\longrightarrow$ mfld ✓



$\longrightarrow$ mfld



not closed.

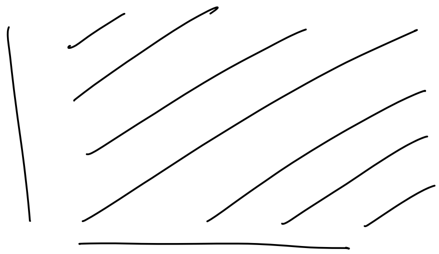
- Group actions

$\mathbb{Z} \circlearrowright \mathbb{R}$

$n \cdot x = x + n$

$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$

$(n, x) \longmapsto (x, x+n)$ $\longrightarrow$ $\mathbb{R}/\mathbb{Z} = S^1$

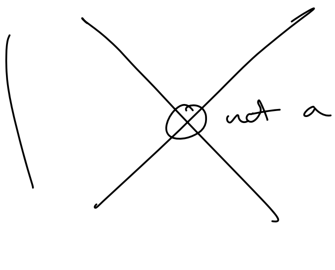


- $\mathbb{Z}/2 \circlearrowright \mathbb{R}$

$\pm 1 \cdot x = \pm x$

$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$

$(\pm 1, x) \longmapsto (x, \pm x)$



not a mfld.

Lecture 4 proper

Vector field on $\mathbb{R}^n$

2 equivalent POV.

① $\tilde{V} : \mathbb{R}^n \to \mathbb{R}^n$

temp notation

Better

② Graph of $\tilde{V}$

$\mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$

$V : x \longmapsto (x, \tilde{V}(x))$

section of $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$\pi \circ V = id_{\mathbb{R}^n}$

Notation

$$\tilde{V}_x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} i^{th}$$ written $\frac{\partial}{\partial x^i}$

means $\tilde{V}(x)$

$\frac{\partial}{\partial x^i}(x) \in x \in \mathbb{R}^n$

so general v.f. is $V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$ fns of x

convention (Einstein) omit Σ. $V = V^i \frac{\partial}{\partial x^i}$

Why? a v.f. gives you a way to diff'te a fn: $f : \mathbb{R}^n \to \mathbb{R}^m$, $V \in \mathrm{Vect}(\mathbb{R}^n)$

$$\underbrace{(Vf)}_{\text{confusing}}{}_x := \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon V) - f(x)}{\varepsilon} \in \mathbb{R}^m$$

(check)

$$= df_x(V_x)$$

so $\left(\frac{\partial}{\partial x^i} f\right)(x) = \frac{\partial f}{\partial x^i}(x)$

Friday Lie bracket $\left[V^i \frac{\partial}{\partial x^i}, W^i \frac{\partial}{\partial x^i}\right] = V^i \left(\frac{\partial W^j}{\partial x^i}\right) \frac{\partial}{\partial x^j} - W^i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial x^j}$

(not on exercises)

* If $\varphi : \underset{\in \mathbb{R}^n}{\mathcal{U}} \to \underset{\in \mathbb{R}}{\hat{\mathcal{U}}}$ is a diffeomorphism, V vect field on $\mathcal{U}$,

then $\varphi_* V$ defines a vector field on $\hat{\mathcal{U}}$ by

$$(\varphi_* V)_q = (q \times) \, d\varphi_{\varphi^{-1}(q)} V_{\varphi^{-1}(q)}$$

Boothby: φ-related

If $S \subseteq \mathbb{R}^n$, a vector field on S is a (smooth) section

$V: S \to S \times \mathbb{R}^n$ s.t. $V(q) \in q \times T_q S$

(Recall $T_q S$ = image $d\psi_{\psi^{-1}(q)}$ for any chart)

If $f: S \to \mathbb{R}$, $V \in T_q S$,

$$(Vf)_x := \lim_{\varepsilon \to 0} \frac{f(x) - f(x+v\varepsilon)}{\varepsilon} \quad \text{for any extension of } f$$

[mfld charts $\leftrightarrow$ Taylor's thm.]

ie df well defined exactly on TV

Generalized: If $\varphi: S \to \tilde{S}$ a diffeomorphism, $V \in \mathrm{Vect}(S)$, $(\varphi_* V)_q := d\varphi_{\varphi^{-1}(q)} V_{\varphi^{-1}(q)}$ is a vector field on $\tilde{S}$.

If ψ is a chart for S, $d\psi_p: \mathbb{R}^n \xrightarrow{\sim} T_p S \to$ every vector field on S is locally $\psi_* V$ for some vector field on $\mathbb{R}^n$.